last time:

La: $B \subseteq \mathbb{R}^n$ bdd, ∂B $(n-1)$-Lipschitz parametr.,

$\Lambda \subseteq \mathbb{R}^n$ lattice

Then $\#(\Lambda \cap aB) = \dfrac{\mu(B)}{vol(\mathbb{R}^n/\Lambda)} a^n + O(a^{n-1})$

$a \geq 1$

Prf: wlog $\Lambda = \mathbb{Z}^n$, $D := [0,1]^n$

$n_0(a) := \#\{x \in \Lambda \mid x + D \subseteq aB\}$

$\wedge\!|$ $\qquad\qquad\qquad \wedge\!|$

$\#(\Lambda \cap aB) \qquad \mu(aB) = a^n \mu(B)$

$\wedge\!| \qquad\qquad\qquad \wedge\!|$

$n_1(a) := \#\{x \in \Lambda \mid x + D \cap aB \neq \emptyset\}$

STP: $|n_1(a) - n_0(a)| = O(a^{n-1})$

Clear: $|n_1(a) - n_0(a)| \leq \#\{x \in \Lambda \mid x + D \cap \partial(aB) \neq \emptyset\}$

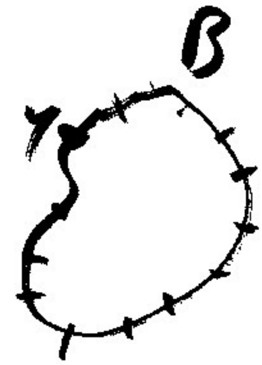
Let $f_1, \dots, f_m : [0,1]^{n-1} \to \mathbb{R}^n$ Lipschitz

with $\bigcup \operatorname{Im} f_i = \partial B$

Pick $c_i > 0$, s.t.

$$\frac{|f_i(x)-f_i(y)|}{|x-y|} < c_i \quad \forall x, y \in [0,1]^{n-1}$$

Set $c := \sqrt{n} \cdot \max\limits_{i=1,\dots,m} \{c_i\}$



Claim: Each $y \in \partial B$ lies within distance c/a to a point in $P := \{ f_i(\frac{r_1}{a}, \dots, \frac{r_{n-1}}{a}) \mid$
$1 \le i \le m, r_i \in \mathbb{Z},$
$0 \le r_1, \dots, r_{n-1} \le a \}$

Pf: $y = f_i(x_1, \dots, x_{n-1})$ for some i,

$x_1, \dots, x_{n-1} \in [0,1]$

Set $r_i = \lfloor a x_i \rfloor \in \mathbb{Z}$, $|x_i - \frac{r_i}{a}| \le 1/a$

$$\Rightarrow \left|y - f_i\left(\frac{r_1}{a}, \dots, \frac{r_{n-1}}{a}\right)\right|$$

$$\le c_i \left|(x_1, \dots, x_{n-1}) - \left(\frac{r_1}{a}, \dots, \frac{r_{n-1}}{a}\right)\right|$$

$$< \frac{c_i}{a}\sqrt{n-1} \le \frac{c}{a}$$

$\Rightarrow$ each pt in $a\cdot\partial B = \partial(aB)$ lies in dist. c from $a\cdot P$

we get: $\#\{r \in \Lambda \mid r + D \cap \partial(aB) \neq \emptyset\}$

$\le \#\{r \in \Lambda \mid r$ has distance $\sqrt{n}$ from $\partial(aB)\}$

$\le \#\{r \in \Lambda \mid r$ has dist $c + \sqrt{n}$ from $a\cdot P\}$

$\le \#\{r \in \Lambda \mid |r| \le c + \sqrt{n}\} \cdot \# aP$

$\le$ const. $m \cdot (a+1)^{n-1}$ $\square$

Next aim: $K/\mathbb{Q}$ finite

$\Rightarrow \{ \mathfrak{p} \subseteq \mathcal{O}_K \text{ prime}, N(\mathfrak{p}) \text{ prime} \}$ is infinite

$\Leftarrow$ ✓ $f(\mathfrak{p} | p) = 1$

($\Leftrightarrow$ inf. many primes $p \in \mathbb{Z}$ split compl. in K)

Idea of proof:

Set $$\zeta_K(s) := \sum_{\substack{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K \\ \text{int. ideal}}} \frac{1}{N(\mathfrak{a})^s}, \quad \operatorname{Re} s > 1$$

"Dedekind zeta fct for K"

If $K = \mathbb{Q} \rightsquigarrow \zeta_K(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

Show 1) $\zeta_K(s)$ has a pole at $s = 1$

2) Divergence depends only

on primes $\mathfrak{p} \subseteq \mathcal{O}_K$ with $N(\mathfrak{p})$ prime ⑤

Note: $\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with

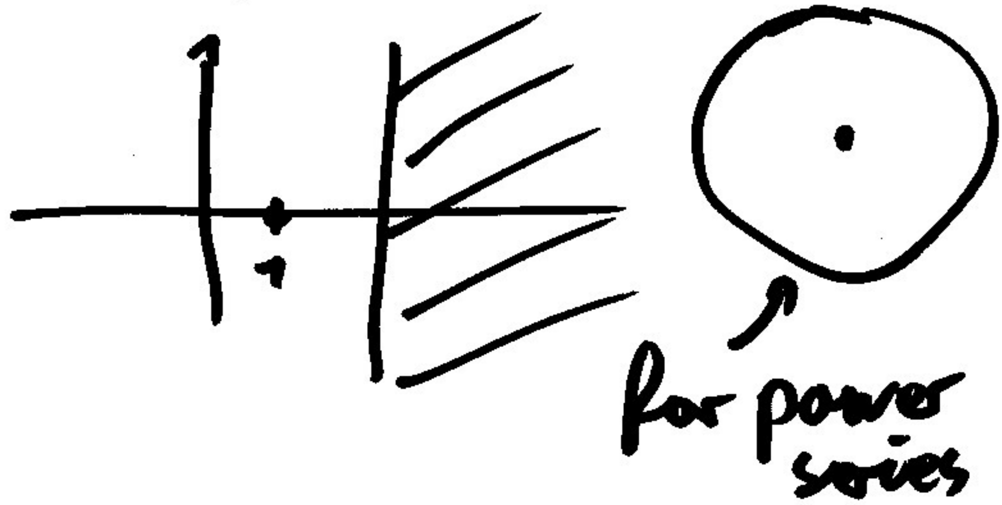
$$a_n = \#\{0 \neq \mathfrak{a} \subseteq \mathcal{O}_K \mid N(\mathfrak{a}) = n\}$$

Def: A Dirichlet series is a (formal) sum $\sum_{n=1}^{\infty} \frac{b_n}{n^s}$ with $b_n \in \mathbb{C}$

Fact: If $f(s) := \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ converges abs. for some

$s_0 \in \mathbb{C}$, $\mathrm{Re}\, s_0 > 0$, then

$f(s)$ conv. for all $s \in \mathbb{C}$, $\mathrm{Re}\, s \geq \mathrm{Re}(s_0)$



Thm: 1) $\zeta_K(s)$ converges abs for $\operatorname{Re} s > 1$

2) $\zeta_K(s)$ can be analytically extended to a meromorphic fct. for $\operatorname{Re} s > 1 - \frac{1}{n}$, $n := [K:\mathbb{Q}]$ with a simple pole at $s = 1$ with residue

$$\frac{2^{r_1}(2\pi)^{r_2} R_K \cdot h}{w \cdot \sqrt{|\Delta_K|}} =: r$$

Prf for 1): $\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n = \#\{\mathfrak{a} \subseteq \mathcal{O}_K \mid N(\mathfrak{a}) = n\}$

$a_n \leq [K:\mathbb{Q}] \cdot \#\{\text{prime factors of } n\}$

$\leq [K:\mathbb{Q}] \cdot \log_2 n$

$\Rightarrow$ $a_n = \mathcal{O}(n^{\varepsilon})$, $\varepsilon > 0$

i.e. $\exists C_\varepsilon > 0$, s.t.

$$\sum_{n=1}^{\infty} \frac{a_n}{|n^s|} = \sum^{\infty} \frac{a_n}{n^{\mathrm{Re}\,s}} \leq C_\varepsilon \cdot \sum_{n=1}^{\infty} \frac{n^{\varepsilon}}{n^{\mathrm{Re}\,s}}$$

$$= C_\varepsilon \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}\,s-\varepsilon}}}_{\substack{\text{conv. for } \mathrm{Re}\,s-\varepsilon>1 \\ \Leftrightarrow\ \mathrm{Re}\,s > 1+\varepsilon}}$$

For 2):

Recall

$$N(t) = \#\{0 \neq \alpha \in \mathcal{O}_K \mid N(\alpha) \leq t\}$$

$$= \sum_{n \leq t} a_n =: r \cdot t + \mathcal{O}(t^{1-\frac{1}{n}})$$

STP general La

La: $f(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$, conv. for Re $s \gg 0$

$S_t := \sum_{n \le t} b_n$ Assume ex. $\varkappa \in \mathbb{C}$, $0 < \delta \le 1$, s.t.

$$S_t = \varkappa \cdot t + O(t^{1-\delta}), \quad t \to \infty$$

$\Rightarrow$ $f(s)$ can be merom. extended for Re $s > 1 - \delta$ with at most one simple pole at $s = 1$ with residue $\varkappa$

Note: E.g. if $S_t = O(1)$

$\Rightarrow$ $f(s)$ extends holom. for $\mathrm{Re}(s) > 0$

First deal $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

$(\rightsquigarrow S_t = t + O(1))$

i.e. $\zeta(s)$ can be ext. for $\operatorname{Re} s > 0$

& one pole at $s = 1$, res. $= 1$

Prf:

$\operatorname{Re} s > 1 \Rightarrow \int_1^{\infty} t^{-s}\,dt = \left[\frac{1}{1-s} \cdot t^{-s+1}\right]_1^{\infty}$

$= \frac{1}{s-1} \qquad (|t^{-s+1}| = t^{-\operatorname{Re} s+1}$

Now, $\zeta(s) - \int_1^{\infty} t^{-s}\,dt$

$$= \sum_{n=1}^{\infty} \int_n^{n+1} \underbrace{\left(\frac{1}{n^s} - \frac{1}{t^s}\right)}_{= \int_{x=n}^{t} s \cdot \frac{1}{x^{s+1}}\,dx} dt$$

$$= \sum_{n=1}^{\infty} \int_{t=n}^{n+1} \int_{x=n}^{t} s \cdot \frac{1}{x^{s+1}}\, dx\, dt$$

$$= \sum_{n=1}^{\infty} \int_{x=n}^{n+1} \int_{t=x}^{n+1} s \frac{1}{x^{s+1}}\, dt\, dx$$

$$= \sum_{n=1}^{\infty} s \int_{n}^{n+1} \frac{n+1-x}{x^{s+1}}\, dx =: g(s)$$

$$\& \left| \sum_{n=1}^{\infty} s \int_{n}^{n+1} \frac{n+1-x}{x^{s+1}}\, dx \right|$$

$$\leq |s| \cdot \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{1}{x^{\mathrm{Re}\, s+1}}\, dx$$

$$= \frac{|s|}{\mathrm{Re}\, s} \Rightarrow g(s) \text{ converges for } \mathrm{Re}\, s > 0$$

$\Rightarrow \zeta(s) = \underbrace{\frac{1}{s-1} + g(s)}$

makes sense for $\operatorname{Re} s > 0$

Back to Ia:

Set $g(s) = f(s) - x \cdot \zeta(s)$

$$= \sum_{n=1}^{\infty} \frac{c_n}{n^s}, \quad c_n = b_n - x$$

Then $S'_t = \sum_{n \leq t} c_n$ satisfies

$$S'_t = O(t^{1-\delta})$$

STP: $g(s)$ analytic cont. for

$\operatorname{Re} s > 1-\delta$

12

For $\operatorname{Re} s \gg 0$

$$g(s) = \sum_{n=1}^{\infty} \frac{S_n' - S_{n-1}'}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{S_n'}{n^s} - \sum_{n=1}^{\infty} \frac{S_n'}{(n+1)^s}$$

$$= \sum_{n=1}^{\infty} S_n' \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right)$$

$$= \sum_{n=1}^{\infty} S_n' \int_n^{n+1} s \cdot t^{-s+1} \, dt$$

Pick $C > 0$, s.t. $|S_t'| \leq C \cdot t^{1-\delta}$.

Let $\sigma = \operatorname{Re} s$

Then

$$\left| \sum_{n=1}^{\infty} S_n' \int_n^{n+1} s \cdot t^{-s+1} \, dt \right| \cancel{\leq C \sum_{n=1}^{\infty} n^{1-\delta}}$$

$$\le C|s|\sum_{n=1}^{\infty} n^{1-\delta}\int_{n}^{n+1} t^{-\sigma-1}\,dt$$

$$\le C|s|\sum_{n=1}^{\infty} \int_{n}^{n+1} \underbrace{t^{1-\delta}\cdot t^{-\sigma-1}}_{t^{-\sigma-\delta}}\,dt$$

$$= C\cdot|s|\,\frac{1}{\sigma+\delta-1} \quad \text{for } \sigma+\delta>1$$

$\Rightarrow$ $g(s)$ holomorphic $\sigma > 1-\delta$.

Lemma: For $\mathrm{Re}\, s > 1$ ← Euler product

$$\zeta_K(s) = \prod_{\substack{0\neq \mathfrak{p}\subseteq \mathcal{O}_K \\ \text{prime}}} \frac{1}{1-N(\mathfrak{p})^{-s}}$$

$$\| $$

$$\sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$$

$$= \prod_{\substack{0\neq \mathfrak{p}\subseteq \mathcal{O}_K \\ \text{prime}}} \Big(\sum_{n=1}^{\infty} \underbrace{N(\mathfrak{p})^{-ns}}_{N(\mathfrak{p}^n)^{-s}} \Big)$$

Prf: STP Euler product converges ($\rightsquigarrow$ use unique fact.) for $\mathrm{Re}\, s > 1$

Recall $\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot x^n$

for $|x| < 1$

(actually for $x \in \mathbb{C}$, $|x| \leq 1$, $x \neq 1$)

$$\log\left(\prod_{\mathfrak{p}} \frac{1}{1-N(\mathfrak{p})^{-\sigma}}\right)$$

$$= \sum_{\mathfrak{p}} \log\left(\frac{1}{1-N(\mathfrak{p})^{-\sigma}}\right)$$

$$\underset{\substack{\uparrow \\ N(\mathfrak{p}) \geq p}}{\leq} [K:\mathbb{Q}] \cdot \sum_{p} \log\left(\frac{1}{1-p^{-\sigma}}\right)$$

$$= [K:\mathbb{Q}] \cdot \sum_{p} \sum_{n=1}^{\infty} \frac{1}{n} p^{-\sigma n}$$

$$\le [K:\mathbb{Q}] \sum_{p} \underbrace{\sum_{n=1}^{\infty} \frac{1}{p^{\sigma n}}}_{\frac{1}{1-p^{-\sigma}} - 1 = \frac{1}{p^{\sigma}-1}}$$

$$\le [K:\mathbb{Q}] \cdot \sum_{p} \underbrace{\frac{1}{p^{\sigma}-1}}_{\le \frac{2}{p^{\sigma}}} \le [K:\mathbb{Q}] \cdot 2 \cdot \underbrace{\sum_{p} \frac{1}{p^{\sigma}}}_{<\infty \text{ if } \sigma > 1} \quad \square$$

Corollary: $\displaystyle\sum_{\substack{0 \neq \mathfrak{p} \subseteq \mathcal{O}_K \\ \text{primes}}} \frac{1}{N(\mathfrak{p})^{s}}$

$$\sim \sum_{\mathfrak{p},\, \deg \mathfrak{p} = 1} \frac{1}{N(\mathfrak{p})^{s}}$$

$$\sim \log \frac{1}{s-1} \quad , \quad s \to 1$$

where $f(s) \sim g(s)$ if $\lim\limits_{s \searrow 1} \frac{f(s)}{g(s)} = 1$

along real axis

$\deg \mathfrak{p}$ is def. by $p^{\deg(\mathfrak{p})} = N(\mathfrak{p})$,

$(p) = \mathbb{Z} \cap \mathfrak{p}$

Prf: $\zeta_K(s) = \varkappa \cdot \frac{1}{s-1} + \underbrace{g(s)}_{\text{hol. at } s=1}$

$$\Rightarrow \frac{\log \zeta_K(s)}{\log\left(\frac{1}{s-1}\right)} = \frac{\log\left(\frac{1}{s-1}\left(\varkappa + (s-1)\cdot g(s)\right)\right)}{\log\left(\frac{1}{s-1}\right)}$$

$$= 1 + \underbrace{\frac{\log(n+(s-1)\cdot g(s))}{\log\left(\frac{1}{s-1}\right)}}_{\to 0,\ s \searrow 1}$$

$$\Longrightarrow \log \zeta_K(s) \sim \log \frac{1}{s-1}$$

For $\operatorname{Re} s > 1$

$$\log \zeta_K(s) = \sum_{\mathfrak{p}} \log \left(\frac{1}{1 - N(\mathfrak{p})^{-s}} \right)$$

$$= \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{1}{n \cdot N(\mathfrak{p})^{ns}}$$

$$= \sum_{\substack{\mathfrak{p}, \deg \mathfrak{p} \\ = 1}} \frac{1}{N(\mathfrak{p})^s} + \underbrace{\sum_{\substack{\mathfrak{p}, \deg \mathfrak{p} \\ \geq 2}} \frac{1}{N(\mathfrak{p})^s} + \sum_{\mathfrak{p}, n \geq 2} \frac{1}{n \cdot N(\mathfrak{p})^{ns}}}_{=: I}$$

Claim: I remains valid for $s > 1$

$$\left(\rightsquigarrow \log \zeta_K(s) \sim \sum_{\mathfrak{p}, \deg \mathfrak{p}} \frac{1}{N(\mathfrak{p})^s}\right)$$

$$\Big|\sum_{\substack{\mathfrak{p}, \deg \mathfrak{p} \\ \geq 2}} \frac{1}{N(\mathfrak{p})^s}\Big| \leq [K:\mathbb{Q}] \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}}}_{\text{conv. for } \sigma > \frac{1}{2}}$$

$$\Big|\sum_{\mathfrak{p}, n \geq 2} \frac{1}{n \cdot N(\mathfrak{p})^{ns}}\Big|$$

$$\leq \sum_{\mathfrak{p}, n \geq 2} \frac{1}{(N(\mathfrak{p})^2)^\sigma} \leq [K:\mathbb{Q}] \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{(n^2)^\sigma}}_{\text{conv. for } \sigma > \frac{1}{2}} < \infty$$

$\square$

Recall:

$$\zeta_K(s) = \varkappa \cdot \frac{1}{s-1} + \underbrace{g(s)}_{\substack{\text{hol. at } s=1 \\ \text{for } \mathrm{Re}\, s > 1-\frac{1}{n}}}$$

$$\& \quad \varkappa = \frac{2^{r_1} (2\pi)^{r_2} R_K \cdot h}{w \sqrt{|\Delta_K|}}$$

Next aim: Get h from $\zeta_K(s)$

Want: Find a way to evaluate $\lim_{s \to 1} (s-1) \cdot \zeta_K(s)$!